ALMOST PRIME TRIPLES AND CHEN'S THEOREM

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ABSTRACT. We show that there are infinitely many primes p such that not only does $p + 2$ have at most two prime factors, but $p + 6$ also has a bounded number of prime divisors. This refines the well known result of Chen [3].

1. INTRODUCTION

The twin prime conjecture states that there are infinitely many primes p such that $p+2$ is also prime. Although the conjecture has resisted our efforts, there has been spectacular partial progress. One well known result is Chen's theorem [3] that there are infinitely many primes such that $p + 2$ has at most two prime factors. In a different direction, building on the work of Goldston, Pintz, and Yıldırım [5], it has recently been shown by Zhang [11] that there are bounded gaps between consecutive primes infinitely often. The numerical result has been improved in the works of the Polymath8 project [9] and Maynard [7], and the bounded gaps result has also been extended to prime tuples by Maynard [7] and Tao (unpublished).

The twin prime conjecture is a special case of the Hardy-Littlewood conjecture, which postulates asymptotics for prime tuples in general. An example is that one expects that the number of primes $p \leq x$ such that $p + 2$ and $p + 6$ are simultaneously prime should be asymptotic to

$$C \frac{x}{\log^3 x}$$

for a certain positive constant C (given by (32)). In this direction, it has been proven that there are infinitely many natural numbers n such that $n(n+2)(n+6)$ is almost prime — that is, $n(n + 2)(n + 6)$ has at most r prime factors, for some finite r. More specifically, Porter [10] proved this statement for $r = 8$ and this was improved by Maynard [8] to $r = 7$.

We are interested in proving an analogue of Chen's theorem for prime tuples. More precisely, we show that there are infinitely many primes p such that $p + 2$ has at most two prime factors, and $p + 6$ has at most r prime factors for some finite r.

Theorem 1. *Let $\pi_{1,2,r}(x)$ denote the number of primes $p \leq x$ such that $p+2$ has at most two prime factors and $p + 6$ has at most r prime factors. Then*

$$\pi_{1,2,r}(x) \gg \frac{x}{\log^3 x} \tag{1}$$

for $r = 76$.

Our basic philosophy, which the proof will illustrate, is the following. Suppose one has polynomials $f_1(x), \ldots, f_{k+1}(x)$ and positive integers $r_1, \ldots, r_k$. Then, if the weighted sieve can prove that

$$f_1(n) = P_{r_1}, \ldots, f_k(n) = P_{r_k}$$

for infinitely many integers n, then one should be able to modify the argument to show the existence of a positive integer r_{k+1} such that

$$f_1(n) = P_{r_1}, \ldots, f_{k+1}(n) = P_{r_{k+1}}$$

for infinitely many integers n.

Our approach uses the weighted sieve which appeared in Chen's original work, as well as the vector sieve of Brüdern and Fouvry [1]. We will also use a Selberg upper bound sieve of "mixed dimension". The value of r in our theorem could be improved by using a more elaborate weighted sieve, but we will not pursue this.

2. The basic setup

In the sequel, p and p_i shall always denote primes. Let $\varepsilon > 0$ be a small positive constant, and let $x^\varepsilon < \xi_2 < \xi_1 \le x^{1/3}$ be parameters to be decided in due course. We will work with the set

$$\mathcal{A} = \{p + 2 : x^{1/3} < p \le x - 6, \, (p + 6, P(\xi_2)) = 1\},$$

where

$$P(w) = \prod_{p < w} p$$

as usual.

The basic idea in Chen's argument is to consider the expression

$$(2) \qquad S_1 = S(\mathcal{A}; \xi_1) - \frac{1}{2} \sum_{\xi_1 \le p \le x^{1/3}} S(\mathcal{A}_p; \xi_1) - \frac{1}{2} N_0,$$

where

$$N_0 = \#\{p_1 p_2 p_3 \in \mathcal{A} : \xi_1 \le p_1 \le x^{1/3} < p_2 < p_3\}.$$

One then has an inequality of the form

$$(3) \qquad S_1 \le \#\mathcal{A}^{(0)} + \#\mathcal{A}^{(1)} + \frac{1}{2}\#\mathcal{A}^{(2)},$$

where

$$\mathcal{A}^{(0)} = \{n \in \mathcal{A} : p_1^2 \mid n \text{ for some } p_1 \ge \xi_1\},$$

$$\mathcal{A}^{(1)} = \{n \in \mathcal{A} : n \text{ prime, or } n = p_1 p_2 \text{ with } p_2 > p_1 > x^{1/3}\},$$

and

$$\mathcal{A}^{(2)} = \{n \in \mathcal{A} : n = p_1 p_2 \text{ with } p_2 > x^{1/3} \ge p_1 \ge \xi_1\}.$$

The bound (3), which the reader may easily verify, is closely related to the inequality used by Halberstam and Richert [6, Chapter 11, (2.1)], for example.

One immediately has $\#\mathcal{A}^{(0)} \ll x/\xi_1$, which will be sufficiently small for our purposes. Moreover one can see that if $n \in \mathcal{A}^{(1)} \cup \mathcal{A}^{(2)}$ then $n = p + 2$ with $n = P_2$ and $n + 4 = P_r$, where $r = [(\log x)/(\log \xi_2)]$. We therefore obtain a result of the type given in our theorem provided that we can give a suitable positive lower bound for S_1. This can be achieved by using the vector sieve of Brüdern and Fouvry [1] in place of the usual upper and lower bound sieves.

There are a number of methods to try to improve the value of r obtained by this naive approach. We choose to include a simple weighted sieve in order to eliminate those triples $(p, p + 2, p + 6)$ for which $p + 6$ has many prime factors. (The reader will observe that one could do better by incorporating more elaborate weights into (2).)

We proceed to define the sets

$$\mathcal{B}^{(i)} = \{p + 6 : p + 2 \in \mathcal{A}^{(i)}\}, \quad (i = 1, 2)$$

and a weight function

$$(4) \qquad w_p = 1 - \frac{\log p}{\log y}$$

where $y = x^{1/v}$ for some positive constant v to be decided in due course. At this stage we insist only that $\xi_2 < y < x$. Since any element of $\mathcal{B}^{(i)}$ is coprime to $P(\xi_2)$, and since $w_p < 0$ for $p > y$, we now have

$$
\begin{aligned}
\sum_{\xi_2 \leq p \leq y} w_p \# \mathcal{B}_p^{(i)} &\geq \sum_{2 \leq p \leq x} w_p \# \mathcal{B}_p^{(i)} \\
&= \sum_{b \in \mathcal{B}^{(i)}} \left(\omega(b) - \frac{1}{\log y} \sum_{p \mid b} \log p \right) \\
&\geq \sum_{b \in \mathcal{B}^{(i)}} \left(\omega(b) - \frac{1}{\log y} \log x \right) \\
&= \sum_{b \in \mathcal{B}^{(i)}} (\omega(b) - v).
\end{aligned}
$$

Here, as usual, $\omega(b)$ denotes the number of distinct prime factors of b. It then follows from (3) that if λ is any positive constant then

$$
\begin{aligned}
S_1 - \lambda & \sum_{\xi_2 \leq p \leq y} w_p \left(\# \mathcal{B}_p^{(1)} + \frac{1}{2} \# \mathcal{B}_p^{(2)} \right) \\
&\leq O(x/\xi_1) + \left(\sum_{b \in \mathcal{B}^{(1)}} + \frac{1}{2} \sum_{b \in \mathcal{B}^{(2)}} \right) (1 + \lambda v - \lambda \omega(b)) \\
&\leq O(x/\xi_1) + (1 + \lambda v) \#\{b \in \mathcal{B}^{(1)} \cup \mathcal{B}^{(2)} : \omega(b) < \lambda^{-1} + v\} \\
(5) \qquad &\leq O(x/\xi_2) + (1 + \lambda v) \#\{b \in \mathcal{B}^{(1)} \cup \mathcal{B}^{(2)} : \omega(b) < \lambda^{-1} + v, \, b \text{ square-free}\}.
\end{aligned}
$$

Here we use the observation the the number of elements of $\mathcal{B}^{(i)}$ which are not square-free must be $O(x/\xi_2)$, since any such element is coprime to $P(\xi_2)$ by definition.

We therefore seek to show that

$$(6) \qquad S_1 - \lambda \sum_{\xi_2 \leq p \leq y} w_p \left(\# \mathcal{B}_p^{(1)} + \frac{1}{2} \# \mathcal{B}_p^{(2)} \right) \geq \{c + o(1)\} \frac{x}{(\log x)^3}$$

for some positive constant c. Substituting our expression for S_1 from (2), we see that we must bound $S(\mathcal{A}; \xi_1)$ from below, which we accomplish using a combination of the linear sieve with the vector sieve. We require upper bounds for the rest of the terms. Here, we use two distinct methods. For

$$\sum_{\xi_1 \leq p \leq x^{1/3}} S(\mathcal{A}_p; \xi_1)$$

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we will use the vector sieve for some ranges of p and the Selberg sieve for other ranges of p. For the remaining terms it turns out to be more efficient to apply the Selberg sieve. Our application has the novel feature that the sieving dimension changes from 2 (for primes $p < \xi_2$) to 1 (for larger primes) part way through the range. Naturally, for the term N_0 we first apply Chen's famous "reversal of rôles" trick before applying the upper bound sieve.

3. SIEVING TOOLS

3.1. The linear sieve.
In the Rosser–Iwaniec linear sieve one has a real parameter $D \geq 2$ and constructs coefficients $\lambda^{\pm}(d)$ supported on the positive integers $d \leq D$ such that

$$\lambda^{\pm}(d) = \mu(d) \text{ or } 0, \text{ for all } d \leq D$$

and

$$\sum_{d|n} \lambda^-(d) \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda^+(d)$$

for all positive integers $n|P(z)$ for some parameter z. Suppose we have a multiplicative function $h(d) \in [0,1)$ such that

$$(7) \qquad \prod_{w \leq p < z} \left(1 - h(p)\right)^{-1} \leq \frac{\log z}{\log w}\left(1 + \frac{L}{\log w}\right)$$

for $z \geq w \geq 2$, for some parameter L. Then, by Theorem 11.12 of Friedlander and Iwaniec [4], we have

$$(8) \qquad \sum_{d|P(z)} \lambda^+(d)h(d) \leq \left\{F(s) + O_L\left((\log D)^{-1/6}\right)\right\} V(z,h) \qquad (s \geq 1)$$

and

$$\sum_{d|P(z)} \lambda^-(d)h(d) \geq \left\{f(s) + O_L\left((\log D)^{-1/6}\right)\right\} V(z,h) \qquad (s \geq 2)$$

where $F(s)$ and $f(s)$ are the standard upper and lower bound functions for the linear sieve, with $s = (\log D)/(\log z)$, and

$$V(z,h) = \prod_{p < z} \left(1 - h(p)\right).$$

Moreover one sees from [4, (6.31)–(6.34)] that

$$(9) \qquad \sum_{d|P(z)} \lambda^-(d)h(d) \leq V(z,h) \leq \sum_{d|P(z)} \lambda^+(d)h(d).$$

3.2. The Fundamental Lemma sieve.
Let $\mathcal{U}$ be a set of positive integers, possibly with multiplicities, and suppose that

$$(10) \qquad \#\mathcal{U}_d = h^*(d)Y + r(d)$$

for some multiplicative function $h^*(d) \in [0,1)$. We assume for simplicity that

$$(11) \qquad h^*(p) \leq C_0 p^{-1},$$

for some constant $C_0 \geq 2$. Then

$$\prod_{w \leq p < z} \left(1 - h^*(p)\right)^{-1} \leq K \left(\frac{\log z}{\log w}\right)^{\kappa}$$

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for $z \geq w \geq 2$ for appropriate constants K and κ depending only on C_0. Hence Corollary 6.10 of Friedlander and Iwaniec [4] applies, and yields

$$S(\mathcal{U}; z) = \{1 + O_{C_0}(e^{-s})\}YV(z, h^*) + O(\sum_{d < z^s} |r(d)|),$$

for $z \geq 2$ and $s \geq 1$.

An inspection of the proof makes it clear that one only uses (10) for values $d \mid P(z)$.

3.3. The vector sieve.

Let $\mathcal{W}$ be a finite subset of $\mathbb{N}^2$. Suppose that $z_1, z_2 \geq 2$ with

$$\log z_1 \asymp \log z_2$$

and write $\boldsymbol{z} = (z_1, z_2)$. For $\boldsymbol{d} = (d_1, d_2)$ and $\mathbf{n} = (n_1, n_2)$, we write $\boldsymbol{d}|\mathbf{n}$ to mean that $d_i|n_i$ for $1 \leq i \leq 2$. Define as usual

$$\mathcal{W}_{\boldsymbol{d}} = \{\mathbf{n} \in \mathcal{W} : \boldsymbol{d}|\mathbf{n}\},$$

and

$$S(\mathcal{W}; \boldsymbol{z}) = \{(m, n) \in \mathcal{W} : (P(z_1), m) = (P(z_2), n) = 1\}.$$

Suppose that

$$\#\mathcal{W}_{\boldsymbol{d}} = h(\boldsymbol{d})X + r(\boldsymbol{d})$$

for some multiplicative function $h(\boldsymbol{d}) \in (0, 1]$ such that $h(p, 1) + h(1, p) - 1 < h(p, p) \leq h(p, 1) + h(1, p)$ for all primes p and

(12) $$h(p, 1), h(1, p) \leq C_1 p^{-1}, \quad \text{and} \quad h(p, p) \leq C_1 p^{-2}$$

for some constant $C_1 \geq 2$. Then using the vector sieve and the linear sieve, we will derive both upper and lower bounds for $S(\mathcal{W}; \boldsymbol{z})$.

For $i = 1, 2$, let λ_i^+ and λ_i^- denote the coefficients of the upper and lower bound linear sieves of level

$$D_i = z_i^{s_i}, \quad (i = 1, 2)$$

where $1 \leq s_i \ll 1$. Further, let $\delta = \mu * 1$, $\delta_i^+ = \lambda_i^+ * 1$ and $\delta_i^- = \lambda_i^- * 1$. Note that $\delta_i^- \leq \delta \leq \delta_i^+$, and that

(13) $$\delta(m)\delta(n) \leq \delta_1^+(m)\delta_2^+(n)$$

and

(14) $$\delta(m)\delta(n) \geq \delta_1^-(m)\delta_2^+(n) + \delta_1^+(m)\delta_2^-(n) - \delta_1^+(m)\delta_2^+(n),$$

for any natural numbers m and n.

In applying the vector sieve we will want to replace $h(\boldsymbol{d})$ by $h_1(d_1)h_2(d_2)$, where

$$h_1(d) = h(d, 1), \quad \text{and} \quad h_2(d) = h(1, d).$$

There is no difficulty when d_1 and d_2 are coprime, but there are potential problems when they share a common factor. We circumvent this issue by using a preliminary application of the Fundamental Lemma sieve. Suppose we are given $z_0 \geq 2$ and positive integers d_1, d_2 coprime to $P(z_0)$. Let $\mathcal{U} = \mathcal{U}(\boldsymbol{d})$ be the set of products mn as (m, n) runs over $\mathcal{W}_{\boldsymbol{d}}$, the values mn being counted according to multiplicity. Then if $d \mid P(z_0)$ we see using the multiplicativity of h that (10) holds with $Y = h(\boldsymbol{d})X$,

$$h^*(d) = \sum_{d = e_1 e_2 e_3} h(e_1 e_3, e_2 e_3)\mu(e_3)$$

and

$$r(d) = \sum_{d = e_1 e_2 e_3} r(d_1 e_1 e_3, d_2 e_2 e_3)\mu(e_3).$$

In particular $h^*(p) = h(p, 1) + h(1, p) - h(p, p) \in [0, 1)$, and (11) holds with suitable $C_0 = 2C_1$. The Fundamental Lemma sieve therefore shows that

$$S\big(\mathcal{W}_{\boldsymbol{d}}; (z_0, z_0)\big) \;=\; S(\mathcal{U}(\boldsymbol{d}), z_0)$$

$$\text{(15)} \qquad\qquad = \; h(\boldsymbol{d}) X V(z_0, h^*) + O(h(\boldsymbol{d}) X e^{-s}) + O\left(\sum_{e_1 e_2 e_3 < z_0^s} |r(d_1 e_1 e_3, d_2 e_2 e_3)| \right).$$

We can now apply the upper bound vector sieve. Suppose that $z_1, z_2 \geq z_0$, and define

$$P(z_0, z) = \prod_{z_0 \leq p < z} p.$$

Let

$$\mathcal{W}^* = \{(m, n) \in \mathcal{W} : \big(mn, P(z_0)\big) = 1\}.$$

Then according to (13) we have

$$S(\mathcal{W}; \boldsymbol{z}) \;=\; \sum_{(m,n) \in \mathcal{W}^*} \delta\big((m, P(z_0, z_1))\big) \delta\big((n, P(z_0, z_2))\big)$$

$$\leq \; \sum_{(m,n) \in \mathcal{W}^*} \left(\sum_{d_1 | (m, P(z_0, z_1))} \lambda_1^+(d_1) \right) \left(\sum_{d_2 | (n, P(z_0, z_2))} \lambda_2^+(d_2) \right)$$

$$= \; \sum_{d_1 | P(z_0, z_1)} \sum_{d_2 | P(z_0, z_2)} \lambda_1^+(d_1) \lambda_2^+(d_2) \# \mathcal{W}_{\boldsymbol{d}}^*.$$

However $\# \mathcal{W}_{\boldsymbol{d}}^* = S\big(\mathcal{W}_{\boldsymbol{d}}; (z_0, z_0)\big)$, whence (15) shows that

$$S(\mathcal{W}; \boldsymbol{z}) \leq X V(z_0, h^*) \Sigma + O(E_1) + O(E_2),$$

where

$$\Sigma = \sum_{d_1 | P(z_0, z_1)} \sum_{d_2 | P(z_0, z_2)} \lambda_1^+(d_1) \lambda_2^+(d_2) h(\boldsymbol{d})$$

and the error terms are

$$E_1 = X e^{-s} \sum_{d_1 < D_1} \sum_{d_2 < D_2} h(\boldsymbol{d})$$

and

$$E_2 = \sum_{f_1 \leq D_1 z_0^s} \sum_{f_2 \leq D_2 z_0^s} \tau^2(f_1) \tau^2(f_2) |r(f_1, f_2)|.$$

(We write $\tau(\ldots)$ for the divisor function as usual.)

To estimate Σ we wish to replace $h(\boldsymbol{d})$ by $h_1(d_1) h_2(d_2)$. These are equal when d_1 and d_2 are coprime. Otherwise we note that

$$\text{(16)} \qquad h(d_1, d_2) \leq C_0^{\omega(d_1 d_2)} (d_1 d_1)^{-1} \ll \tau(d_1)^{C_0} \tau(d_2)^{C_0} (d_1 d_2)^{-1}$$

by (12), and similarly $h_1(d_1) \ll \tau(d_1)^{C_0} d_1^{-1}$ and $h_2(d_2) \ll \tau(d_2)^{C_0} d_2^{-1}$. Hence if d_1 and d_2 are not coprime then

$$h(d_1, d_2) = h(d_1, 1) h(1, d_2) + O(\tau(d_1)^{C_0} \tau(d_2)^{C_0} (d_1 d_2)^{-1}).$$

This latter case will only hold if there is a prime $p \geq z_0$ which divides both d_1 and d_2. As a result we may deduce that

$$\Sigma = \sum_{d_1 | P(z_0, z_1)} \sum_{d_2 | P(z_0, z_2)} \lambda_1^+(d_1) \lambda_2^+(d_2) h_1(d_1) h_2(d_2)$$

$$+ O\left(\sum_{p \geq z_0} \sum_{e_1 < D_1/p} \sum_{e_2 < D_2/p} \tau(pe_1)^{C_0} \tau(pe_2)^{C_0} (p^2 e_1 e_2)^{-1} \right).$$

The leading term factors as

$$\left\{ \sum_{d_1 | P(z_0, z_1)} \lambda_1^+(d_1) h_1(d_1) \right\} \left\{ \sum_{d_2 | P(z_0, z_2)} \lambda_2^+(d_2) h_2(d_2) \right\}$$

and so if h_1 and h_2 satisfy the condition (7) the inequalities (8) and (9) will lead to an upper bound

$$\left\{ F(s_1) + O_L\left((\log z_1)^{-1/6} \right) \right\} \left\{ F(s_2) + O_L\left((\log z_2)^{-1/6} \right) \right\} V_1 V_2,$$

with

$$V_i = \prod_{z_0 \leq p < z_i} (1 - h_i(p)), \quad (i = 1, 2).$$

The error term is

$$\ll \sum_{p \geq z_0} p^{-2} (\log z_1)^{2C_0} (\log z_2)^{2C_0} \ll z_0^{-1} (\log z_1 z_2)^{2+C_0}.$$

Hence if we take

$$z_0 = \exp(\sqrt[3]{\log z_1 z_2})$$

then we find that

$$\Sigma \leq F(s_1) F(s_2) V_1 V_2 \{ 1 + O\left((\log z_1 z_2)^{-1/6} \right) \}$$

on observing that $V_i \gg (\log z_1 z_2)^{-1}$, by (7).

The error term E_1 is easily handled using (16). This produces

$$E_1 \ll X e^{-s} (\log z_1 z_2)^{2+C_0} \ll X \exp\{ -(\log z_1 z_2)^{1/4} \}$$

on choosing

$$s = \sqrt[3]{\log z_1 z_2}.$$

The bound (11) shows that $V(z_0, h^*) \gg (\log z_0)^{-C_0}$, whence we may conclude that

$$E_1 \ll X V(z_0, h^*) V_1 V_2 F(s_1) F(s_2) (\log z_1 z_2)^{-1/6}.$$

Moreover if we write $D = D_1 D_2$ we have

$$E_2 \ll_\varepsilon \sum_{d_1 < D_1 (z_1 z_2)^\varepsilon} \sum_{d_2 < D_2 (z_1 z_2)^\varepsilon} \tau(d_1 d_2)^4 |r(d_1, d_2)| \ll_\varepsilon \sum_{d_1 d_2 < D^{1+\varepsilon}} \tau(d_1 d_2)^4 |r(d_1, d_2)|$$

for any fixed $\varepsilon > 0$.

We can therefore summarize our result as the first statement in the following proposition.

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Proposition 1. *Suppose that $h(\boldsymbol{d})$ satisfies (12) and that $h_1(d)$ and $h_2(d)$ both satisfy (7). Assume further that $D = z_1^{s_1} z_2^{s_2}$ with $1 \leq s_1, s_2 \ll 1$, and $\log z_1 \asymp \log z_2$ as in our setup of the Vector Sieve. Then*

$$S(\mathcal{W}; \boldsymbol{z}) \;\leq\; XV(z_0, h^*)V_1 V_2 F(s_1)F(s_2)\{1 + O\left((\log D)^{-1/6}\right)\}$$

$$+ O_\varepsilon \left(\sum_{d_1 d_2 < D^{1+\varepsilon}} \tau(d_1 d_2)^4 |r(d_1, d_2)| \right)$$

for any fixed $\varepsilon > 0$. Indeed if we write

$$\sigma_i = \frac{\log D}{\log z_i}, \qquad (i = 1, 2)$$

we may replace $F(s_1)F(s_2)$ by

$$F(\sigma_1, \sigma_2) := \inf\{F(s_1)F(s_2) : s_1/\sigma_1 + s_2/\sigma_2 = 1, \; s_i \geq 1 \, (i = 1, 2)\}.$$

Similarly we have

$$S(\mathcal{W}; \boldsymbol{z}) \;\geq\; XV(z_0, h^*)V_1 V_2 f(\sigma_1, \sigma_2)\{1 + O\left((\log D)^{-1/6}\right)\}$$

$$+ O_\varepsilon \left(\sum_{d_1 d_2 < D^{1+\varepsilon}} \tau(d_1 d_2)^4 |r(d_1, d_2)| \right)$$

for any fixed $\varepsilon > 0$, where

$$f(\sigma_1, \sigma_2) : \;=\; \sup\{f(s_1)F(s_2) + f(s_2)F(s_1) - F(s_1)F(s_2) : s_1/\sigma_1 + s_2/\sigma_2 = 1,$$
$$s_i \geq 2 \, (i = 1, 2)\}.$$

The lower bound is proved along the same lines as the upper bound, using (14) in place of (13). In handling the expression corresponding to Σ we encounter a leading term of the form

$$\Sigma_1^- \Sigma_2^+ + \Sigma_2^- \Sigma_1^+ - \Sigma_1^+ \Sigma_2^+,$$

where

$$\Sigma_i^\pm = \sum_{d | P(z_0, z_i)} \lambda_i^+(d) h_i(d).$$

In general, if

(17) $$U_i \geq \Sigma_i^+ \geq L_i \geq 0 \quad \text{and} \quad \Sigma_i^- \geq L_i \quad \text{for} \quad i = 1, 2$$

then

$$\Sigma_1^- \Sigma_2^+ + \Sigma_2^- \Sigma_1^+ - \Sigma_1^+ \Sigma_2^+ \;\geq\; L_1 \Sigma_2^+ + L_2 \Sigma_1^+ - \Sigma_1^+ \Sigma_2^+$$
$$= \; L_1 L_2 - (\Sigma_1^+ - L_1)(\Sigma_2^+ - L_2).$$

Since $\Sigma_2^+ - L_2 \geq 0$ and $U_1 - L_1 \geq 0$ the above expression is at least

$$L_1 L_2 - (U_1 - L_1)(\Sigma_2^+ - L_2) \geq L_1 L_2 - (U_1 - L_1)(U_2 - L_2) = L_1 U_2 + L_2 U_1 - U_1 U_2.$$

To complete the proof of the proposition we apply the above inequality with

$$U_i = \left\{ F(s_i) + O_L\left((\log D_i)^{-1/6}\right) \right\} V_i \quad \text{and} \quad L_i = \left\{ f(s_i) + O_L\left((\log D_i)^{-1/6}\right) \right\} V_i,$$

the required inequalities (17) following from our description of the linear sieve, given in subsection 3.1, and noting that $\log D_i \asymp \log D$ for $i = 1, 2$.

3.4. **Selberg's sieve.** Let $\mathcal{W}$ be a set of positive integers and for each prime $p < z$ let $\Omega(p)$ be a set of residue classes modulo p. We would like to estimate

$$S(\mathcal{W}; z) = \#\{w \in \mathcal{W} : w \notin \Omega(p) \text{ for all } p < z\}$$

using Selberg's sieve.

Suppose that

(18) $$\#\{n \in \mathcal{W} : [n \bmod p] \in \Omega(p) \text{ if } p \mid d \text{ and } p < z\} = h(d)X + r_d$$

for some multiplicative function $h(d) \in [0, 1)$. Then the usual analysis of Selberg's sieve (see Halberstam and Richert [6, Theorem 3.2], for example) shows that

$$S(\mathcal{W}; z) \leq \frac{X}{G(z)} + \sum_{d < z^2} 3^{\omega(d)} |r_d|,$$

in which

$$G(z) = \sum_{d < z} \mu^2(d) g(d)$$

where g is the multiplicative function supported on squarefree numbers defined by

$$g(p) = \frac{h(p)}{1 - h(p)}.$$

For our applications we will have

(19) $$ph(p) = \begin{cases} 2 + O(p^{-1}), & p < z_2, \\ 1 + O(p^{-1}), & z_2 \leq p < z_1, \\ 0, & \text{otherwise,} \end{cases}$$

where $2 \leq z_2 < z_1 \leq z$.

We now need to develop the asymptotics for $G(z)$.

Proposition 2. *Preserve notation as above, and define*

$$s_i = \frac{\log z}{\log z_i}.$$

Let

$$\rho : (0, \infty) \to \mathbb{R}$$

be Dickman's function, defined by

$$\rho(s) = \begin{cases} 0, & s \leq 0, \\ 1, & 0 < s \leq 1, \end{cases}$$

and

(20) $$s\rho'(s) = -\rho(s - 1)$$

for $s > 1$. Further let

$$B : (0, \infty)^2 \to \mathbb{R}$$

be defined by

(21) $$B(s_1, s_2)^{-1} = e^{-2\gamma} \int\!\!\int_{\{(w_1.w_2):w_1/s_1 + w_2/s_2 \leq 1\}} \rho(w_1)\rho(w_2) dw_1 dw_2.$$

Then we have that

(22) $$G(z)^{-1} \sim B(s_1, s_2)V(z, h)$$

if $1 \leq s_1, s_2 \ll 1$.

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We delay the proof of this result until §5. Note that the level of distribution required will be $D = z^2$, and that we have taken $s_i = (\log D)/(2 \log z_i)$, rather than the more normal $s_i = (\log D)/(\log z_i)$. It is easy to translate to the latter notation, but the definition of $B(s_1, s_2)$ would look rather less natural.

3.5. A version of the Bombieri–Vinogradov Theorem.
In the previous sub-sections we introduced remainder terms which can be bounded in our applications by using a suitable version of the Bombieri–Vinogradov Theorem. We begin by stating a convenient result from the literature.

Lemma 1. *For $z_1, z_2, ..., z_r \geq 2$, define the set with multiplicities*

$$(23) \qquad P(z_1, ..., z_r) = \{p^{(r)} = p_1...p_r : p_1 \geq z_1, ..., p_r \geq z_r\}.$$

Let $\pi_r(x; q, a)$ be the number of $p^{(r)} \in P(z_1, ..., z_r)$ such that $p^{(r)} \equiv a \pmod{q}$ and $p^{(r)} \leq x$. Further let $\pi_r(x; q)$ be the number of $p^{(r)} \in P(z_1, ..., z_r)$ such that $p^{(r)} \leq x$ and $(p^{(r)}, q) = 1$. Then for any $A > 0$ there exists $B = B(A) > 0$ such that

$$(24) \qquad \sum_{q < x^{1/2}(\log x)^{-B}} \max_{(a,q)=1} \left| \pi_r(x; q, a) - \frac{1}{\phi(q)} \pi_r(x; q) \right| \ll x(\log x)^{-A},$$

where the implied constant depends only on r and A.

This is Theorem 22.3 of Friedlander and Iwaniec [4].

Note that the result reduces to the classical version of the Bombieri–Vinogradov Theorem when $r = 1$. In our applications we will sometimes need to replace the set defined in (23) with sets of the form

$$(25) \qquad P(z_1, ..., z_r, y_1, ..., y_r) = \{p^{(r)} = p_1...p_r : y_1 \geq p_1 \geq z_1, ..., y_r \geq p_r \geq z_r\}$$

for $z_1, ...z_r, y_1, ..., y_r \geq 2$ where we allow $y_i = \infty$ in which case the condition $y_i \geq p_i$ is automatically fulfilled. The lemma clearly holds for these sets as well since we may express a set of the form (25) in terms of sets of the form (23), using the inclusion-exclusion principle.

We will actually need the following slightly different version of the above lemma.

Lemma 2. *Let $P(z_1, ..., z_r, y_1, ..., y_r)$ be as in (25) and fix notation as in Lemma 1. For each $q \geq 1$, let*

$$(26) \qquad R_q(x) = \max_{(a,q)=1} \left| \pi_r(x; q, a) - \frac{1}{\phi(q)} \pi_r(x; q) \right|.$$

Then for any $A > 0$ and $k \geq 1$ there exists $B = B(A, k) > 0$ such that

$$(27) \qquad \sum_{q < x^{1/2}(\log x)^{-B}} \tau(q)^k R_q(x) \ll x(\log x)^{-A},$$

where the implied constant depends only on r, k and A.

Proof. For $q < x$ we have

$$(28) \qquad R_q(x) \ll \frac{x}{\phi(q)},$$

so that

$$(29) \qquad \sum_{q < x} \tau(q)^{2k} R_q(x) \ll x(\log x)^{2^{2k}}.$$

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On the other hand, Lemma 1 shows that for any $A' > 0$ we will have

$$(30) \qquad \sum_{q < Q} R_q(x) \ll x (\log x)^{-A'}$$

for $Q = x^{1/2} (\log x)^{-B'(A')}$. The result then follows from (29) and (30) by applying the Cauchy Schwarz inequality to (27), and choosing A' sufficiently large in terms of A and k. $\qquad \square$

4. PROOF OF THE THEOREM

4.1. Bounding $S(\mathcal{A}; \xi_1)$. We now take

$$\xi_i = x^{\theta_i} \quad (i = 1, 2) \quad \text{and} \quad y = x^{\theta},$$

where θ_1, θ_2 and θ are constants satisfying

$$0 < \theta_2 < \theta_1 < \frac{1}{3} \quad \text{and} \quad \theta_2 < \theta < 1.$$

Thus $\theta = v^{-1}$ in the notation of §2.

We will apply the lower bound vector sieve to the set

$$\mathcal{W} = \{(p + 2, p + 6) : x^{1/3} < p \le x - 6\},$$

taking $\boldsymbol{z} = (\xi_1, \xi_2)$ and $X = \pi(x)$. Since $p > 6$ we see that we cannot have $d_1 \mid p + 2$ and $d_2 \mid p + 6$ unless $(d_1, d_2) = (d_1, 2) = (d_2, 6) = 1$. We therefore set

$$h(\boldsymbol{d}) = \begin{cases} \frac{1}{\phi(d_1 d_2)}, & \text{if } (d_1, d_2) = (d_1, 2) = (d_2, 6) = 1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\#\mathcal{W}_{\boldsymbol{d}} = h(\boldsymbol{d}) \pi(x) + R(\boldsymbol{d}),$$

whence Lemma 2 gives

$$(31) \qquad \sum_{\substack{\boldsymbol{d} \\ d_1 d_2 < x^{1/2 - \varepsilon}}} \tau(d_1 d_2)^4 |R(\boldsymbol{d})| \ll x (\log x)^{-A},$$

for any positive constant A.

We now use the vector sieve lower bound from Proposition 1. According to (31), the remainder sum can be bounded adequately when $D = x^{1/2 - 2\epsilon}$. The Euler factors in $V(z_0, h^*) V_1 V_2$ are

$$1 = 4\left(1 - \tfrac{1}{2}\right)^2 \quad \text{for} \quad p = 2,$$

$$1 - \tfrac{1}{2} = \tfrac{9}{8}\left(1 - \tfrac{1}{3}\right)^2 \quad \text{for} \quad p = 3,$$

$$1 - \frac{2}{p - 1} = \left(1 - \frac{3p - 1}{(p - 1)^3}\right)\left(1 - \frac{1}{p}\right)^2 \quad \text{for} \quad 5 \le p < z_0,$$

$$\left(1 - \frac{1}{p - 1}\right)^2 = \left(1 - \frac{1}{(p - 1)^2}\right)^2 \left(1 - \frac{1}{p}\right)^2 \quad \text{for} \quad z_0 \le p < \xi_2,$$

and

$$\left(1 - \frac{1}{p - 1}\right) = \left(1 - \frac{1}{(p - 1)^2}\right)\left(1 - \frac{1}{p}\right) \quad \text{for} \quad \xi_2 \le p < \xi_1,$$

whence

$$V(z_0, h^*) V_1 V_2 \sim C V(\xi_1) V(\xi_2)$$

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with

$$(32) \qquad C = \frac{9}{2} \prod_{p>3} \left(1 - \frac{3p-1}{(p-1)^3}\right)$$

and

$$V(z) = \prod_{p<z}(1 - p^{-1}) \sim \frac{e^{-\gamma}}{\log z}.$$

Note that C is the constant appearing in the Hardy-Littlewood conjectures for such prime tuples. We therefore obtain the lower bound

$$(33) \qquad S(\mathcal{A}; \xi_1) \geq (C + o(1))\pi(x)V(\xi_1)V(\xi_2)f\left((2\theta_1)^{-1}, (2\theta_2)^{-1}\right).$$

4.2. **The terms $S(\mathcal{A}_p; \xi_1)$.** We may apply the upper bound vector sieve with the same set $\mathcal{W}$ as before, noting that $h(pd_1, d_2) = \phi(p)^{-1}h(\boldsymbol{d})$ when $d_1 \mid P(\xi_1)$, $d_2|P(\xi_2)$ and $p \geq \xi_1$. This easily leads to the bound

$$
\begin{aligned}
S(\mathcal{A}_p; \xi_1) \quad \leq \quad & (C + o(1))\frac{\pi(x)}{p-1}V(\xi_1)V(\xi_2)F(s_1(p), s_2(p)) \\
& + O_\varepsilon \left(\sum_{d_1 d_2 < p^{-1}D^{1+\varepsilon}} \tau(d_1 d_2)^4 |R(pd_1, d_2)|\right),
\end{aligned}
$$

with

$$s_i(p) = \frac{\log D/p}{\log \xi_i} \quad (i = 1, 2).$$

Hence for any $P \leq P'$ we have

$$
\begin{aligned}
\sum_{P \leq p \leq P'} S(\mathcal{A}_p; \xi_1) \quad \leq \quad & (C + o(1))\pi(x)\left(\sum_{P \leq p \leq P'}\frac{F(s_1(p), s_2(p))}{p-1}\right)V(\xi_1)V(\xi_2) \\
& + O_\varepsilon \left(\sum_{pd_1 d_2 < D^{1+\varepsilon}} \tau(d_1 d_2)^4 |R(pd_1, d_2)|\right).
\end{aligned}
$$

The remainder sum is negligible, by Lemma 2, if $D = x^{1/2-2\varepsilon}$. It follows that

$$(34) \qquad \sum_{P \leq p \leq P'} S(\mathcal{A}_p; \xi_1) \leq (C + o(1))\pi(x)V(\xi_1)V(\xi_2)\int_P^{P'} \frac{F(\sigma_1(t), \sigma_2(t))}{t \log t}dt,$$

where we now have

$$(35) \qquad \sigma_i(t) = \frac{\log \sqrt{x}/t}{\log \xi_i} \quad (i = 1, 2).$$

Alternatively we can use Selberg's sieve as in subsection 3.4. For a given prime p we take $\mathcal{W} = \mathcal{W}^{(p)}$ to consist of the values $(q + 2)(q + 6)$ where q runs over primes in the interval $x^{1/3} < q \leq x - 6$ such that $p \mid q + 2$. For each prime r we use the residue classes

$$(36) \qquad \Omega(r) = \begin{cases} \emptyset, & r = 2, \\ \{-2\}, & r = 3, \\ \{-2, -6\} & 5 \leq r < \xi_2, \\ \{-2\}, & \xi_2 \leq r < \xi_1 \\ \emptyset, & r \geq \xi_1. \end{cases}$$

It is natural to take $X = \pi(x)/(p-1)$ and

$$(37) \qquad h(r) = \begin{cases} 0, & r = 2, \\ \frac{1}{2}, & r = 3 \\ \frac{2}{r-1}, & 5 \le r < \xi_2, \\ \frac{1}{r-1}, & \xi_2 \le r < \xi_1, \\ 0, & r \ge \xi_1. \end{cases}$$

Let $z \ge \xi_1$ and write

$$s_i = \frac{\log z}{\log \xi_i} \quad i = 1, 2$$

Then

$$V(z, h) \sim C V(\xi_1) V(\xi_2).$$

If we write $r_d^{(p)}$ for the corresponding remainder in (18) we will have

$$\begin{aligned} S(\mathcal{A}_p; \xi_1) &= S(\mathcal{W}; z) \\ &\le \frac{\pi(x)}{(p-1)G(z)} + \sum_{d < z^2} 3^{\omega(d)} |r_d^{(p)}| \\ &= (C + o(1)) \frac{\pi(x)}{p-1} B(s_1, s_2) V(\xi_1) V(\xi_2) + \sum_{d < z^2} 3^{\omega(d)} |r_d^{(p)}|. \end{aligned}$$

Moreover

$$|r_d^{(p)}| \le \tau(d) \max_{(a, pd) = 1} \left| \{\pi(x - 6; pd, a) - \pi(x^{1/3}; pd, a)\} - \frac{\pi(x - 6) - \pi(x^{1/3})}{\phi(q)} \right|.$$

Lemma 2 then shows that if we choose $z = (\sqrt{x}/P)^{1/2} (\log x)^{-C_0}$ with a suitably large constant C_0 then

$$\sum_{P \le p \le P'} S(\mathcal{A}_p; \xi_1) \le (C + o(1)) \pi(x) \left(\sum_{P \le p \le P'} \frac{B(\tilde{\sigma}_1(p), \tilde{\sigma}_2(p))}{p - 1} \right) V(\xi_1) V(\xi_2),$$

with

$$\tilde{\sigma}_i(t) = \frac{\log \left(\frac{\sqrt{x}}{t} \right)^{1/2}}{\log \xi_i},$$

provided that $\xi_1 \le z$. We then deduce that

$$\sum_{P \le p \le P'} S(\mathcal{A}_p; \xi_1) \le (C + o(1)) \pi(x) V(\xi_1) V(\xi_2) \int_P^{P'} \frac{B(\tilde{\sigma}_1(t), \tilde{\sigma}_2(t))}{t \log t} dt.$$

Comparison with (34) now shows that

$$(38) \qquad \sum_{\xi_1 \le p \le x^{1/3}} S(\mathcal{A}_p, \xi_1) \le (C + o(1)) \pi(x) V(\xi_1) V(\xi_2) I(\theta_1, \theta_2),$$

with

$$(39) \qquad I(\theta_1, \theta_2) = \int_{\theta_1}^{1/3} \alpha^{-1} \min \left\{ F\left(\frac{1 - 2\alpha}{2\theta_1}, \frac{1 - 2\alpha}{2\theta_2} \right), B\left(\frac{1 - 2\alpha}{4\theta_1}, \frac{1 - 2\alpha}{4\theta_2} \right) \right\} d\alpha.$$

4.3. Estimating N_0 via Chen's rôle-reversal trick.

The number N_0 is defined in terms of products $p_1 p_2 p_2 \in \mathcal{A}$. However we can change our point of view and write

$$N_0 = \#\{p + 2 \in \mathcal{B} : p \text{ prime}\},$$

where

$$\mathcal{B} = \{p_1 p_2 p_3 : \xi_1 \le p_1 \le x^{1/3} < p_2 < p_3, \ x^{1/3} + 2 < p_1 p_2 p_3 \le x - 4,$$
$$(p_1 p_2 p_3 + 4, P(\xi_2)) = 1\}.$$

Thus instead of sieving numbers $p + 2$ and $p + 6$ we will sieve numbers $p_1 p_2 p_3 - 2$ and $p_1 p_2 p_3 + 4$. This is Chen's reversal of rôles. Following the approach of subsection 3.4, we let

$$\mathcal{W} = \{p_1 p_2 p_3 : \xi_1 \le p_1 \le x^{1/3} < p_2 < p_3, \text{ and } p_1 p_2 p_3 \le x - 4\}$$

and for each prime r we define the set $\Omega(r)$ by

$$(40) \qquad \Omega(r) = \begin{cases} \emptyset, & r = 2, \\ \{2\}, & r = 3, \\ \{2, -4\} & 5 \le r < \xi_2, \\ \{2\}, & \xi_2 \le r < z \\ \emptyset, & r \ge z. \end{cases}$$

It follows that

$$N_0 \le S(\mathcal{W}; z)$$

for any z between ξ_2 and $x^{1/4}$, say.

It is natural to take $X = \#\mathcal{W}$ and to choose the function $h(r)$ to be given by (37) as before, except that now $h(r) = 1/(r - 1)$ for $\xi_2 \le r < z$. With this definition we will have

$$V(z, h) \sim CV(z)V(\xi_2) \sim CV(\xi_1)V(\xi_2)s_1^{-1},$$

where $s_1 = (\log z)/(\log \xi_1)$. Moreover if we define r_d via (18) then we can use Lemma 2 with $z = x^{1/4}(\log x)^{-B/2}$ to show that

$$(41) \qquad \sum_{d < z^2} 3^{\omega(d)} |r_d| \ll x(\log x)^{-A}.$$

In order to do this we replace $\mathcal{W}$ by the set

$$\mathcal{W}_0 = \{p_1 p_2 p_3 : \xi_1 \le p_1 \le x^{1/3} < p_2, p_3, \text{ and } p_1 p_2 p_3 \le x - 4\},$$

to which Lemma 2 applies directly. We should also note that

$$\pi_r(x - 4; q) = \#\mathcal{W}_0 + O(x\xi_1^{-1}) = \#\mathcal{W}_0 + O(x(\log x)^{-A-1}),$$

on allowing for possible common factors of q and $p_1 p_2 p_3$. The error term here is certainly small enough for (41).

We also need to estimate $\#\mathcal{W}$. We find that

$$\#\mathcal{W} \sim \sum_{\substack{\xi_1 \le p_1 \le x^{1/3} \\ x^{1/3} \le p_2 \le (x/p_1)^{1/2}}} \frac{x}{p_1 p_2 \log \frac{x}{p_1 p_2}} \sim x \int_{\xi_1}^{x^{1/3}} \int_{x^{1/3}}^{(x/v)^{1/2}} \frac{du\, dv}{(u \log u)(v \log v) \log\left(\frac{x}{uv}\right)}$$

$$(42) \qquad \qquad \qquad \qquad \sim \pi(x) L(\theta_1^{-1})$$

where

$$(43) \qquad L(s) = \int_{1/s}^{1/3} \int_{1/3}^{\frac{1-\beta}{2}} \frac{d\alpha\, d\beta}{\alpha\beta(1 - \alpha - \beta)},$$

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by the change of variables $u = x^\alpha$ and $v = x^\beta$. We therefore conclude that

$$(44) \qquad N_0 \leq \left(C + o(1)\right)\pi(x)V(\xi_1)V(\xi_2)L\left(\theta_1^{-1}\right)4\theta_1 B\left(1, (4\theta_2)^{-1}\right).$$

It is possible as well to apply the vector sieve here, but the bound (44) is always superior for our application.

4.4. The weighted sieve terms.
We now turn our attention to

$$(45) \qquad T := \sum_{\xi_2 \leq q \leq y} w_q \left(\#\mathcal{B}_q^{(1)} + \frac{1}{2}\#\mathcal{B}_q^{(2)} \right).$$

We write

$$(46) \qquad T \leq \sum_{\xi_2 \leq q \leq y} w_q \left(\#\mathcal{V}_q^{(1)} + \frac{1}{2}\#\mathcal{V}_q^{(2)} \right),$$

where

$$(47) \qquad \mathcal{V}^{(1)} = \{n + 4 : n \in \mathcal{A} \text{ and } (n, P(x^{1/4})) = 1\},$$

and

$$(48) \qquad \mathcal{V}^{(2)} = \{n + 4 : n \in \mathcal{A} \text{ and } n = p_1 p_2, \xi_1 \leq p_1 \leq x^{1/4} < p_2\}.$$

Note that $\mathcal{B}^{(1)} \subset \mathcal{V}^{(1)}$ and every element of $B^{(2)}$ is in $\mathcal{V}^{(2)}$ with the exception of those $n + 4$ with $n \in \mathcal{A}$, $n = p_1 p_2$ with $p_1, p_2 > x^{1/4}$, and those are counted in $\mathcal{V}^{(1)}$.

We begin by examining $\#\mathcal{V}_q^{(1)}$. We will use the Selberg sieve as in subsection 3.4. To be precise, we take

$$\mathcal{W} = \{p \in (x^{1/3}, x - 6] : q \mid p + 6\}$$

and

$$\Omega(r) = \begin{cases} \emptyset, & r = 2 \text{ or } q, \\ \{-2\}, & r = 3, \\ \{-2, -6\} & 5 \leq r < \xi_2, \\ \{-2\}, & \xi_2 \leq r < z, r \neq q. \end{cases}$$

For some $z \in [\xi_2, x^{1/4}]$, we choose

$$X = \frac{1}{q - 1}\#\{p : x^{1/3} < p \leq x - 6\}$$

and use h given by

$$(49) \qquad h(r) = \begin{cases} 0, & r = 2 \text{ or } q, \\ \frac{1}{2}, & r = 3 \\ \frac{2}{r-1}, & 5 \leq r < \xi_2, \\ \frac{1}{r-1}, & \xi_2 \leq r < z, r \neq q, \end{cases}$$

whence

$$V(z, h) \sim CV(z)V(\xi_2).$$

In estimating the individual terms in

$$\sum_{P < q \leq P'} \#\mathcal{V}_q^{(1)},$$

Lemma 2 will allow us to use $z = x^{1/4}q^{-1/2}(\log x)^{-B/2}$, provided that $z \geq \xi_2$. We therefore conclude that

$$\sum_{P < q \leq P'} \#\mathcal{V}_q^{(1)} \leq (C + o(1))\pi(x) \sum_{P < q \leq P'} \frac{1}{q-1} V(z)V(\xi_2)B\left(1, \frac{\log z}{\log \xi_2}\right).$$

Since $\log z \sim \log(x^{1/4}q^{-1/2})$ we have

$$V(z) \sim V(\xi_1)\frac{\log \xi_1}{\log(x^{1/4}q^{-1/2})},$$

and we deduce that

$$\sum_{\xi_2 \leq q \leq y} w_q \#\mathcal{V}_q^{(1)} \leq (C + o(1))\pi(x)V(\xi_1)V(\xi_2)\Sigma,$$

with

$$\Sigma = \sum_{\xi_2 \leq q \leq y} \frac{w_q}{q-1} \frac{\log \xi_1}{\log(x^{1/4}q^{-1/2})} B\left(1, \frac{\log x^{1/4}q^{-1/2}}{\log \xi_2}\right).$$

In order to ensure that $z \geq \xi_2$, we impose the condition that $2\theta_2 + \theta < 1/2$. Bearing in mind the definition (4) of the weights $w_q = 1 - \frac{\log q}{\log y}$ we apply the Prime Number Theorem to see that

$$\Sigma \sim \int_{\xi_2}^y \frac{\log \xi_1}{\log(x^{1/4}q^{-1/2})}\left(1 - \frac{\log t}{\log y}\right) B\left(1, \frac{\log(x^{1/4}t^{-1/2})}{\log \xi_2}\right) \frac{dt}{t \log t}$$

$$\sim \int_{\theta_2}^\theta \frac{4\theta_1}{1 - 2\alpha} \frac{\theta - \alpha}{\alpha\theta} B\left(1, \frac{1 - 2\alpha}{4\theta_2}\right) d\alpha.$$

For notational convenience, let

(50) $$J(\theta_1, \theta_2, \theta) := \int_{\theta_2}^\theta \frac{4\theta_1}{1 - 2\alpha} \frac{\theta - \alpha}{\alpha\theta} B\left(1, \frac{1 - 2\alpha}{4\theta_2}\right) d\alpha,$$

so that

(51) $$\sum_{\xi_2 \leq q \leq y} w_q \#\mathcal{V}_q^{(1)} \leq (C + o(1))J(\theta_1, \theta_2, \theta)\pi(x)V(\xi_1)V(\xi_2),$$

if $2\theta_2 + \theta < 1/2$.

As in the previous section, here too we could have used the vector sieve upper bound, but again the Selberg method is superior.

We now examine $\#\mathcal{V}_q^{(2)}$. Again, we will use the Selberg sieve as in subsection 3.4, but our approach to $\mathcal{V}_q^{(1)}$ and our approach to $\mathcal{V}_q^{(2)}$ differ. In our treatment of $\mathcal{V}_q^{(1)}$, we took $\mathcal{W}$ to be a set of primes p and used the sieve to handle the conditions that $(p+2, P(x^{1/4})) = 1$ and $(p+6, P(\xi_2)) = 1$. Here, we will take $\mathcal{W}$ to be a set of numbers of the form $n = p_1p_2$ for p_1 and p_2 prime, and use the sieve to handle the condition that $n - 2$ is prime and $(n + 4, P(\xi_2)) = 1$

To be precise, we take

$$\mathcal{W} = \{p_1p_2 \in (x^{1/3} + 2, x - 4] : q \mid p_1p_2 + 4, \xi_1 \leq p_1 \leq x^{1/4} < p_2\}$$

and

$$\Omega(r) = \begin{cases} \emptyset, & r = 2 \text{ or } q, \\ \{2\}, & r = 3, \\ \{2, -4\} & 5 \le r < \xi_2, \\ \{2\}, & \xi_2 \le r < z, \, r \ne q. \end{cases}$$

We choose

$$X = \frac{1}{q-1} \#\{p_1 p_2 \in (x^{1/3} + 2, x - 4] : \xi_1 \le p_1 \le x^{1/4} < p_2\}$$

and use h given by (49) as before. Recall that

$$V(z, h) \sim CV(z)V(\xi_2).$$

We have

$$X \sim \frac{1}{q-1} \sum_{\xi_1 \le p_1 \le x^{1/4}} \frac{x}{p_1 \log\left(\frac{x}{p_1}\right)}$$

$$\sim \frac{1}{q-1} \int_{\xi_1}^{x^{1/4}} \frac{x}{t \log\left(\frac{x}{t}\right)} \frac{dt}{\log t}$$

$$\sim \frac{1}{q-1} \pi(x) \int_{\theta_1}^{1/4} \frac{du}{u(1-u)}$$

$$= \frac{1}{q-1} \pi(x) \left(\log \frac{1-\theta_1}{3\theta_1}\right).$$

In estimating

$$\sum_{P < q \le P'} \#\mathcal{V}_q^{(2)},$$

Lemma 2 will again allow us to use $z = x^{1/4} q^{-1/2} (\log x)^{-B/2}$, provided that $z \ge \xi_2$. We therefore conclude that

$$\sum_{P < q \le P'} \#\mathcal{V}_q^{(2)} \le \big(C + o(1)\big) \pi(x) \log\left(\frac{1-\theta_1}{3\theta_1}\right) \sum_{P < q \le P'} \frac{1}{q-1} V(z) V(\xi_2) B\left(1, \frac{\log z}{\log \xi_2}\right).$$

Continuing as in the previous section, we have

$$(52) \qquad \sum_{\xi_2 \le q \le y} w_q \#\mathcal{V}_q^{(2)} \le \big(C + o(1)\big) \log\left(\frac{1-\theta_1}{3\theta_1}\right) J(\theta_1, \theta_2, \theta) \pi(x) V(\xi_1) V(\xi_2),$$

where J is as defined in (50). As in the previous section, here too we could have used the vector sieve upper bound, but again the Selberg method is superior.

4.5. Summary.

Putting (33), (38) and (44) into (2) and by (51), (46) and (52), we have

$$(53) \quad S_1 - \lambda \sum_{\xi_2 \le p \le y} w_p \left(\#\mathcal{B}_p^{(1)} + \frac{1}{2} \#\mathcal{B}_p^{(2)}\right) \ge C\pi(x) V(\xi_1) V(\xi_2) H(\theta_1, \theta_2, \theta, \lambda)(1 + o(1)),$$

where

$$H(\theta_1, \theta_2, \theta, \lambda) = f((2\theta_1)^{-1}, (2\theta_2)^{-1}) - \frac{1}{2} I(\theta_1, \theta_2) - 2L(\theta_1^{-1})\theta_1 B(1, (4\theta_2)^{-1})$$

$$(54) \qquad - \lambda \left(1 + \frac{1}{2} \log \frac{1-\theta_1}{3\theta_1}\right) J(\theta_1, \theta_2, \theta).$$

We chose $\theta_1 = 1/11$, $\theta_2 = 1/410$, $\theta = 1/30$ and $\lambda = 0.0145$. Recall that f is defined as a supremum - using Matlab to conduct a rough search for the sup, we compute that

$$f((2\theta_1)^{-1}, (2\theta_2)^{-1}) \geq 0.9992523...$$

When calculating $I(\theta_1, \theta_2)$, the quantity

$$\min\left\{ F\Big(\frac{1-2\alpha}{2\theta_1}, \frac{1-2\alpha}{2\theta_2}\Big), B\Big(\frac{1-2\alpha}{4\theta_1}, \frac{1-2\alpha}{4\theta_2}\Big) \right\}$$

appears, arising from use of both the vector sieve and our version of Selberg's sieve. For our values of θ_1 and θ_2, $F\big(\frac{1-2\alpha}{2\theta_1}, \frac{1-2\alpha}{2\theta_2}\big)$ is smaller for small values of α, while $B\big(\frac{1-2\alpha}{4\theta_1}, \frac{1-2\alpha}{4\theta_2}\big)$ becomes a better choice at around $\alpha = 0.26$. Again, a Matlab computation with a rough optimization of the value of $F\big(\frac{1-2\alpha}{2\theta_1}, \frac{1-2\alpha}{2\theta_2}\big)$, which was defined as an inf, gives that

$$I(\theta_1, \theta_2) \leq 1.5630111...$$

It is simple to calculate that

$$L(\theta_1^{-1}) = 0.5477550...$$

Further, using that

(55)
$$\int_0^\infty \rho(u)du = \int_0^\infty u\rho(u)du = e^\gamma,$$

we have that

$$B(1,v)^{-1} = e^{-2\gamma} \int_0^\infty \rho(y) \int_0^{1-y/v} 1\,dx\,dy - \int_v^\infty \rho(y) \int_0^{1-y/v} 1\,dx\,dy.$$

On the other hand, using the crude upper bound $\rho(n) \leq \frac{1}{n!}$, we have that

$$\int_v^\infty \rho(y) \int_0^{1-y/v} 1\,dx\,dy \leq \sum_{n \geq \lfloor v \rfloor v} \frac{1}{n!} \leq \frac{e}{\lfloor v \rfloor !},$$

while

$$e^{-2\gamma} \int_0^\infty \rho(y) \int_0^{1-y/v} 1\,dx\,dy = \left(1 - \frac{1}{v}\right)e^{-\gamma}.$$

From this, it follows that

$$B(1, (4\theta_2)^{-1}) = 1.7986199...$$

Substituting the above estimate for $B(1,v)$ into the definition of $J(\theta_1, \theta_2, \theta)$, we find that

$$J(\theta_1, \theta_2, \theta) = 1.1235270...$$

We find that $H(\theta_1, \theta_2, \theta, \lambda) > 0$ for $\lambda < 0.0214$. From (5) this gives a bound for r of the form $r \leq 1/\theta + 1/\lambda < 77$, giving the result that there are infinitely many primes p such that $p + 2$ has at most 2 prime factors and $p + 6$ has at most 76 prime factors.

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5. The average of multiplicative functions appearing in Selberg's sieve

We end by proving Proposition 2. For $i \in \{1, 2\}$, let

$$\chi_i(n) = \begin{cases} 1, & \text{if } p|n \Rightarrow p < z_i, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that g is the multiplicative function supported on squarefree numbers defined by

$$g(p) = \frac{h(p)}{1 - h(p)}.$$

We further define the multiplicative functions k and j by

$$n\mu^2(n)g(n) = (\chi_1 * k)(n) = (\chi_1 * \chi_2 * j)(n),$$

for all natural numbers n, so that

$$\sum_{n \geq 1} \frac{n\mu^2(n)g(n)}{n^s} = \left(\sum_{n \geq 1} \frac{\chi_1(n)}{n^s} \right) \left(\sum_{n \geq 1} \frac{k(n)}{n^s} \right)$$

$$= \left(\sum_{n \geq 1} \frac{\chi_1(n)}{n^s} \right) \left(\sum_{n \geq 1} \frac{\chi_2(n)}{n^s} \right) \left(\sum_{n \geq 1} \frac{j(n)}{n^s} \right).$$

The Dirichlet series above clearly converge for Re $s > 1$. Moreover we see that

$$\sum_{r \geq 1} \frac{j(r)}{r^s} = \prod_p \left(1 + \frac{g(p)}{p^{s-1}} \right) \prod_{p < z_1} \left(1 - \frac{1}{p^s} \right) \prod_{p < z_2} \left(1 - \frac{1}{p^s} \right).$$

Thus

$$j(p) \ll p^{-1}, \qquad \text{and} \qquad j(p^e) \ll 1 \ (e \geq 2).$$

Similarly we find that

$$k(p) = \begin{cases} 1 + O(p^{-1}), & p < z_2, \\ O(p^{-1}), & z_2 \leq p < z_1, \\ 0, & p \geq z_1, \end{cases} \qquad \text{and} \qquad k(p^e) \ll 1 \ (e \geq 2).$$

These estimates suffice to show that

(56)
$$\sum_{R < r \leq 2R} |j(r)| \ll_\varepsilon R^{1/2 + \varepsilon}$$

for any fixed $\varepsilon > 0$, and

(57)
$$\sum_{M < m \leq 2M} |k(m)| \ll M.$$

In order to study $G(z)$ we will first examine the average of $d\mu^2(d)g(d)$, whose behaviour resembles that of $\chi_1 * \chi_2$. We intend to take advantage of the fact that χ_1 and χ_2 are indicator functions of smooth numbers, and the computation of their averages are standard results. We have

(58)
$$\sum_{n \leq x} n\mu^2(n)g(n) = \sum_{m \leq x} k(m)\Psi(\frac{x}{m}; z_1),$$

where $\Psi(x; y)$ is the number of y-smooth numbers below x. It follows from a result of de Bruijn [2] that

$$\Psi(x; y) = x\rho\left(\frac{\log x}{\log y} \right) + O\left(\frac{x}{\log(2x)} \right)$$

uniformly for $1 \leq y \leq x$, where ρ is Dickman's function defined as in the statement of Proposition 2. Continuing from (58), we have

$$\sum_{n \leq x} n \mu^2(n) g(n) = x \sum_{m \leq x} \frac{k(m)}{m} \rho \left(\frac{\log x/m}{\log z_1} \right) + O \left(x \sum_{m \leq x} \frac{|k(m)|}{m \log(2x/m)} \right)$$

(59)
$$= x \sum_{m \leq x} \frac{k(m)}{m} \rho \left(\frac{\log x/m}{\log z_1} \right) + O(x \log \log x),$$

upon observing that

$$\sum_{m \leq x} \frac{|k(m)|}{m \log(2x/m)} \ll \log \log x$$

by (57). A similar calculation yields

$$\sum_{m \leq y} k(m) = \sum_{r \leq y} j(r) \Psi \left(\frac{y}{r}; z_2 \right)$$

$$= y \sum_{r \leq y} \frac{j(r)}{r} \rho \left(\frac{\log y/r}{\log z_2} \right) + O \left(y \sum_{r \leq y} \frac{|j(r)|}{r \log(2y/r)} \right)$$

(60)
$$= y \sum_{r \leq y} \frac{j(r)}{r} \rho \left(\frac{\log y/r}{\log z_2} \right) + O(y(\log 2y)^{-1}),$$

after noting that

$$\sum_{r \leq y} \frac{|j(r)|}{r \log(2y/r)} \ll (\log 2y)^{-1}$$

by (56). Another application of (56) shows that

$$\sum_{r \leq y} \frac{j(r)}{r} \rho \left(\frac{\log y/r}{\log z_2} \right) = \sum_{r \leq \sqrt{y}} \frac{j(r)}{r} \rho \left(\frac{\log y/r}{\log z_2} \right) + O(y^{-1/8}).$$

Since $\rho'(t) \ll t^{-1}$ for $t > 0$ we have $\rho \left(\frac{\log y/r}{\log z_2} \right) = \rho \left(\frac{\log y}{\log z_2} \right) + O \left(\frac{\log r}{\log 2y} \right)$ for $r \leq \sqrt{y}$, whence two more applications of (56) yield

$$\sum_{r \leq \sqrt{y}} \frac{j(r)}{r} \rho \left(\frac{\log y/r}{\log z_2} \right) = \rho \left(\frac{\log y}{\log z_2} \right) \sum_{r \leq \sqrt{y}} \frac{j(r)}{r} + O \left((\log 2y)^{-1} \sum_{r \leq \sqrt{y}} \frac{|j(r)| \log r}{r} \right)$$

$$= \rho \left(\frac{\log y}{\log z_2} \right) \sum_{r=1}^{\infty} \frac{j(r)}{r} + O((\log 2y)^{-1}).$$

It therefore follows from (60) that

$$\sum_{m \leq y} k(m) = C_0 y \rho \left(\frac{\log y}{\log z_2} \right) + O(y(\log 2y)^{-1}),$$

where

(61)
$$C_0 = \sum_{r=1}^{\infty} \frac{j(r)}{r} = V(z_1) V(z_2) V(z,h)^{-1} \sim e^{-2\gamma} (\log z_1)^{-1} (\log z_2)^{-1} V(z,h)^{-1}.$$

Note here that $C_0 \ll 1$, since C_0 can be written as a product of Euler factors each of which is $1 + O(p^{-2})$.

We may now insert the above formula into (59), using partial summation to deduce that

$$
\sum_{n \leq x} n\mu^2(n)g(n) = x \sum_{m \leq x} \frac{k(m)}{m} \rho\left(\frac{\log x/m}{\log z_1}\right) + O(x \log \log x)
$$

$$
= C_0 x \int_1^x \frac{1}{t} \frac{d}{dt}\left\{ t\rho\left(\frac{\log t}{\log z_2}\right)\right\} \rho\left(\frac{\log x/t}{\log z_1}\right) dt + O(x \log \log x).
$$

The integral is

$$
\int_1^x \frac{1}{t}\rho\left(\frac{\log t}{\log z_2}\right)\rho\left(\frac{\log x/t}{\log z_1}\right) dt + \int_1^x \frac{1}{t \log z_2}\rho'\left(\frac{\log t}{\log z_2}\right)\rho\left(\frac{\log x/t}{\log z_1}\right) dt.
$$

However $\rho'(s) = 0$ for $0 < s \leq 1$ and $\rho'(s) \ll s^{-1}$ otherwise. Thus the second integral above is $O(\log \log x)$ so that

$$
\sum_{n \leq x} n\mu^2(n)g(n) = C_0 x \int_1^x \rho\left(\frac{\log t}{\log z_2}\right)\rho\left(\frac{\log x/t}{\log z_1}\right) \frac{dt}{t} + O(x \log \log x).
$$

A further summation by parts now shows that

$$
G(z) = \sum_{n < z} \mu^2(n)g(n)
$$

$$
= C_0 \left\{ \int_1^z \rho\left(\frac{\log t}{\log z_2}\right)\rho\left(\frac{\log z/t}{\log z_1}\right) \frac{dt}{t} + \int_1^z \frac{1}{x} \int_1^x \rho\left(\frac{\log t}{\log z_2}\right)\rho\left(\frac{\log x/t}{\log z_1}\right) \frac{dt}{t} dx \right\}
$$
$$
+ O((\log z)(\log \log z)).
$$

The first integral above is $O(\log z)$, which may be absorbed into the error term, while the second is

$$
(\log z_1)(\log z_2) \int \int_{\{(w_1, w_2): w_1, w_2 \geq 0,\, w_1/s_1 + w_2/s_2 \leq 1\}} \rho(w_1)\rho(w_2) dw_1 dw_2.
$$

The proposition now follows from (61).

References

[1] J. Brüdern and E. Fouvry, Lagrange's four squares theorem with almost prime variables. J. Reine Angew. Math. 454 (1994), 59–96.

[2] N. G. de Bruijn, On the number of positive integers $\leq x$ and free of prime factors $> y$. Nederl. Akad. Wetensch. Proc. Ser. A 54 (1951), 50–60.

[3] J.R. Chen, On the representation of a larger even integer as the sum of a prime and the product of at most two primes. Sci. Sinica 16 (1973), 157–176.

[4] J. Friedlander and H. Iwaniec, *Opera de cribro*. American Mathematical Society Colloquium Publications, 57. American Mathematical Society, Providence, RI, 2010.

[5] D. A. Goldston, J Pintz and C.Y. Yıldırım, Primes in tuples. I. Ann. of Math. (2) 170 (2009), no. 2, 819–862.

[6] H. Halberstam and H.-E. Richert, *Sieve methods.* London Mathematical Society Monographs, No. 4. Academic Press, London-New York, 1974.

[7] J. Maynard, Small gaps between primes. Ann. of Math. (2) 181 (2015), no. 1, 383–413.

[8] J. Maynard, 3-tuples have at most 7 prime factors infinitely often. Math. Proc. Cambridge Philos. Soc. 155 (2013), no. 3, 443–457.

[9] D. H. J. Polymath, New equidistribution estimates of Zhang type. Algebra and Number Theory. 8:9 (2014)

[10] J.W. Porter, Some numerical results in the Selberg sieve method. Acta Arith. 20 (1972), 417–421.

[11] Y. Zhang, Bounded gaps between primes, Ann. of Math. (2) 179 (2014), 1121–1174.

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